

UNIQUENESS OF POSITIVE SOLUTION FOR A QUASILINEAR ELLIPTIC EQUATION IN HEISENBERG GROUP

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ABSTRACT. In this article we are interested in addressing the question of existence and uniqueness of positive solution to a quasilinear elliptic equation involving p-laplacian in Heisenberg Group. The idea is to prove the uniqueness by using Diaz-Saá Inequality in Heisenberg Group which we obtain via a generalized version of Picone's Identity.

1. INTRODUCTION

In this article we will be dealing with the problem of existence and uniqueness of the p-sub-laplacian operator in Heisenberg Group. We begin by recalling the well-known paper of H.Brézis and L.Oswald [2], where the necessary and sufficient condition for the existence and uniqueness of positive solutions were obtained for the equation:

$$-\Delta u = f(x, u) \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega$$

when Ω is a bounded open domain in \mathbb{R}^n .

Almost immediately the result was extended in J.Diáz and J.Saá [3] to the p-laplacian case by introducing a new inequality which came to be later known as the Díaz-Saá Inequality.

The purpose of this paper is to extend the result of [3] in the context of Heisenberg Group, which we will denote by \mathcal{H}^n .

Consider the problem:

$$\begin{aligned} -\Delta_p u &= f(x, u) \text{ in } \Omega \\ u &\geq 0 \text{ and } u \not\equiv 0 \text{ in } \Omega \\ u &= 0 \text{ on } \partial\Omega \end{aligned} \tag{1.1}$$

where Ω is an open bounded domain of \mathcal{H}^n and $1 < p < \infty$.

We consider $f : \Omega \times [0, \infty) \rightarrow (0, \infty)$ satisfying the following hypothesis:

- I The function $r \rightarrow f(x, r)$ is continuous on $[0, \infty)$ for a.e $x \in \Omega$ and for every $r \geq 0$, the function $x \rightarrow f(x, r)$ is in $L^\infty(\Omega)$.
- II The function $r \rightarrow \frac{f(x, r)}{r^{p-1}}$ is strictly decreasing on $(0, \infty)$ for a.e $x \in \Omega$.
- III $\exists C > 0$ s.t $f(x, r) \leq C(r^{p-1} + 1)$ for a.e $x \in \Omega$ and for all r .

We aim to establish the existence and uniqueness of the weak solution to (1.1).

Before we start with our results let us briefly recall some basic notions regarding the Heisenberg Group \mathcal{H}^n along with some literature which is available on the study of Elliptic Equation on Heisenberg Group.

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The Heisenberg Group $\mathcal{H}^n = (\mathcal{R}^{2n+1}, \cdot)$ is Nilpotent Lie Group endowed with the group structure:

$$(x, y, t) \cdot (x', y', t') = (x + x', y + y', t + t' + 2(\langle y, x' \rangle - \langle x, y' \rangle))$$

where $x, y, x', y' \in \mathcal{R}^n$, $t, t' \in \mathcal{R}$ and $\langle \cdot, \cdot \rangle$ denotes the standard inner product in \mathcal{R}^n .

The left invariant vector field generating the Lie algebra is given by

$$\mathcal{T} = \frac{\partial}{\partial t}, \mathcal{X}_i = \frac{\partial}{\partial x_i} + 2y_i \frac{\partial}{\partial t}, \mathcal{Y}_i = \frac{\partial}{\partial y_i} - 2x_i \frac{\partial}{\partial t}, i = 1, 2, \dots, n.$$

and satisfy the following relationship:

$$[\mathcal{X}_i, \mathcal{Y}_i] = -4\delta_{ij}T, [\mathcal{X}_i, \mathcal{X}_j] = [\mathcal{Y}_i, \mathcal{Y}_j] = [\mathcal{X}_i, \mathcal{T}] = [\mathcal{Y}_i, \mathcal{T}] = 0.$$

The generalized gradient is given by $\nabla_H = (\mathcal{X}_1, \mathcal{X}_2, \dots, \mathcal{X}_n, \mathcal{Y}_1, \mathcal{Y}_2, \dots, \mathcal{Y}_n)$. Hence the sub-laplacian Δ_H and the p-sub-Laplacian $\Delta_{H,p}$ are denoted by

$$\Delta_H = \sum_{i=1}^n \mathcal{X}_i^2 + \mathcal{Y}_i^2 = \nabla_H \cdot \nabla_H, \text{ and}$$

$$\Delta_{H,p} = \nabla_H(|\nabla_H|^{p-2} \nabla_H), p > 1$$

We also denote the space $D^{1,p}(\Omega)$ and $D_0^{1,p}(\Omega)$ as $\{u : \Omega \rightarrow \mathcal{R}; u, |\nabla_H u| \in L^p(\Omega)\}$ and the closure of $C_0^\infty(\Omega)$ with respect to the norm $\|u\|_{D_0^{1,p}(\Omega)} = (\int_\Omega |\nabla_H u|^p dx dy dt)^{\frac{1}{p}}$ respectively.

For more details about Heisenberg Group the reader may consult [4].

Some results on the Laplacian and the p-Laplacian has been generalized to the Heisenberg Group with various degree of success. Consider the following problem:

$$\begin{aligned} -\Delta_{H,p} u &= f(u) \text{ in } \Omega \\ u &= 0 \text{ on } \partial\Omega \end{aligned} \tag{1.2}$$

Some of the very first results obtained regarding the above problem for $p = 2$ is by Garofalo-Lanconelli [9], where existence and nonexistence results were derived using integral identities of Rellich-Pohozaev type. In Birindelli et al [10], Liouville theorems for semilinear equations are proved. One can also find monotonicity and symmetry results in Birindelli and Prajapat [8]. As for the p-sub-Laplacian case, Niu et al [7] considered the question of non-uniqueness of the (1.2) using the Picone Identity and the Pohozaev Identities for the p-sub-Laplacian on Heisenberg Group. Results on p-sub-Laplacian involving singular indefinite weight can be found in J.Dou [11] and J.Tyagi [1] and the reference therein.

One of the biggest problem when dealing with p-sub-Laplacian is the non-availability of the $C^{1,\alpha}$ regularity, although it has been proved in Marchi [12] to exist for p near 2. It is worth mentioning that the methods of Díaz-Sáa [3] can't be directly applied here due to the non-availability of $C^{1,\alpha}$ regularity in the Heisenberg Group. In this work we bypass that problem by using a generalized version of Díaz-Sáa Inequality in Heisenberg Group.

2. PRELIMINARY RESULTS

We start this section with the generalized Picone's Identity for p -sub-Laplacian in Heisenberg Group, which is extension of the main result in Euclidean space obtained in [14].

In what follows we will assume $g : (0, \infty) \rightarrow (0, \infty)$ is a locally Lipchitz function that satisfies the differential inequality:

$$g'(x) \geq (p-1)[g(x)]^{\frac{p-2}{p-1}} \text{ a.e in } (0, \infty) \quad (2.1)$$

Remark 2.1. *Example of functions satisfying (2.1) is $g(x) = x^{p-1}$ (where the equality holds) and $e^{(p-1)x}$.*

In what follows we will use ∇ to denote ∇_H and Δ_p to denote $\Delta_{H,p}$.

Theorem 2.2. (Generalized Picone Identity) *Let $1 < p < \infty$ and Ω be any domain in \mathcal{H}^n . Let u and v be differentiable functions on Ω with $v > 0$ a.e in Ω . Also assume g satisfies (2.1). Define*

$$L(u, v) = |\nabla u|^p - p \frac{|u|^{p-2}u}{g(v)} \nabla u \cdot \nabla v |\nabla v|^{p-2} + \frac{g'(v)|u|^p}{[g(v)]^2} |\nabla v|^p \text{ a.e in } \Omega.$$

$$R(u, v) = |\nabla u|^p - \nabla \left(\frac{|u|^p}{g(v)} \right) \cdot \nabla v |\nabla v|^{p-2} \text{ a.e in } \Omega.$$

Then $L(u, v) = R(u, v) \geq 0$. Moreover $L(u, v) = 0$ a.e. in Ω if and only if $\nabla(\frac{u}{v}) = 0$ a.e. in Ω .

Remark 2.3. *Note that there is no restriction on the sign of u , as one can find in Proposition 3 of [13]. When $g(x) = x^{p-1}$ and $u \geq 0$, we get back Lemma 2.1 (i.e, Picone Identity) of [11].*

Proof of Theorem 2.2. Expanding $\nabla(\frac{|u|^p}{g(v)})$ we have,

$$\begin{aligned} \nabla \left(\frac{|u|^p}{g(v)} \right) &= \frac{pg(v)|u|^{p-2}u \nabla u - g'(v)|u|^p \nabla v}{[g(v)]^2} \\ &= p \frac{|u|^{p-2}u \nabla u}{g(v)} - \frac{g'(v)|u|^p \nabla v}{[g(v)]^2}. \end{aligned}$$

Plugging it in $R(u, v)$ we have $R(u, v) = L(u, v)$.

To show positivity of $L(u, v)$ we proceed as follows,

$$\frac{|u|^{p-2}u}{g(v)} \nabla u \cdot \nabla v |\nabla v|^{p-2} \leq \frac{|u|^{p-1}}{g(v)} |\nabla v|^{p-1} |\nabla u| \quad (2.2)$$

and by Young's Inequality we have,

$$p \frac{|u|^{p-1}}{g(v)} |\nabla v|^{p-1} |\nabla u| \leq |\nabla u|^p + (p-1) \frac{|u|^p |\nabla v|^p}{[g(v)]^{\frac{p}{p-1}}} \quad (2.3)$$

Using (2.2) and (2.3) we have,

$$L(u, v) \geq -(p-1) \frac{|u|^p |\nabla v|^p}{[g(v)]^{\frac{p}{p-1}}} + \frac{g'(v)|u|^p}{[g(v)]^2} |\nabla v|^p$$

Now since g satisfies (2.1) i.e, $g'(x) \geq (p-1)[g(x)]^{\frac{p-2}{p-1}}$ we have, $L(u, v) \geq 0$.

Equality holds when the following occurs simultaneously:

$$g'(x) = (p-1)[g(x)]^{\frac{p-2}{p-1}} \quad (2.4)$$

$$\frac{|u|^{p-2}u}{g(v)} \nabla u \cdot \nabla v |\nabla v|^{p-2} = \frac{|u|^{p-1}}{g(v)} |\nabla v|^{p-1} |\nabla u|. \quad (2.5)$$

and,

$$|\nabla u| = \frac{|u\nabla v|}{g(v)^{\frac{1}{p-1}}} \quad (2.6)$$

Set,

$$\mathcal{X} = \{x \in \Omega : \frac{|u\nabla v|}{g(v)^{\frac{1}{p-1}}} = 0\}$$

By equation (2.6) we have,

$$\frac{|u\nabla v|}{g(v)^{\frac{1}{p-1}}} = |\nabla u| = 0 \text{ a.e on } \mathcal{X}. \quad (2.7)$$

from (2.7) and (2.4) we have for $g(x) = x^{p-1}$,

$$\frac{u}{v}\nabla v = \nabla u = 0 \text{ a.e on } \mathcal{X}. \quad (2.8)$$

On \mathcal{X}^c , let

$$w = \frac{|\nabla u|[g(v)]^{\frac{1}{p-1}}}{|u\nabla v|}$$

Hence from the fact that $L(u, v) = 0$ a.e in Ω we have,

$$w^p - pw + p - 1 = 0$$

which holds iff $w = 1$.

Again taking into account (2.4) is true for $g(x) = x^{p-1}$ we have,

$$\nabla u \cdot (\nabla u - \nabla v \frac{u}{v}) = 0 \text{ a.e in } \mathcal{X}^c \quad (2.9)$$

Combining (2.8) and (2.9) we can easily conclude that $L(u, v) = 0$ iff $\nabla(\frac{u}{v}) = 0$ a.e in Ω . \square

With the generalized Picone's Identity in our hands we can now proceed to prove the Picone's Inequality which is the vital ingredient for the proof of Díaz-Saá Inequality. We will present a non-linear version of the Inequality and will closely follow the proof of Abdellaoui-Peral [15].

Theorem 2.4. (Generalized Picone Inequality) *Let $p > 1$ and Ω be a bounded domain in \mathcal{H}^n . If $u, v \in D_0^{1,p}(\Omega)$ s.t $-\Delta_p v = \mu$ where μ is a positive bounded Radon measure with $v|_{\partial\Omega} = 0$, $v(\not\equiv 0) \geq 0$ and g satisfies (2.1). Then we have,*

$$\int_{\Omega} |\nabla u|^p \geq \int_{\Omega} \left(\frac{|u|^p}{g(v)} \right) (-\Delta_p v).$$

Remark 2.5. *When $g(u) = u^{p-1}$, we get Picone's Inequality in Heisenberg Group in Dou [11].*

Before we proceed with the proof of our theorem we need the following lemma:

Lemma 2.6. *Let $p > 1$ and Ω be any domain in \mathcal{H}^n and let $v \in D^{1,p}(\Omega)$ be such that $v \geq \delta > 0$. Then for all $u \in C_c^\infty(\Omega)$ we have,*

$$\int_{\Omega} |\nabla u|^p \geq \int_{\Omega} \left(\frac{|u|^p}{g(v)} \right) (-\Delta_p v).$$

Proof. Since $v \in D^{1,p}(\Omega)$, we can by Meyers-Serrin Theorem, choose $v_n \in C^1(\Omega)$ such that the following holds:

$$v_n > \frac{\delta}{2} \text{ in } \Omega, \quad v_n \rightarrow v \text{ in } \Omega \text{ and } v_n \rightarrow v \text{ a.e in } \Omega.$$

Employing Theorem 2.2 with v_n and u we have,

$$\int_{\Omega} R(u, v_n) = 0 \text{ since } R(u, v_n) = 0 \text{ a.e in } \Omega \text{ and } \forall n \in \mathbb{N}.$$

i.e,

$$\int_{\Omega} |\nabla u|^p \geq \int_{\Omega} \nabla \left(\frac{|u|^p}{g(v_n)} \right) |\nabla v_n|^{p-2} \nabla v_n = \int_{\Omega} \frac{|u|^p}{g(v_n)} (-\Delta_p v_n).$$

Note that since $-\Delta_p$ is a continuous function from $D^{1,p}(\Omega)$ to $D^{-1,p'}(\Omega)$ for $p' = \frac{p}{p-1}$, we have $-\Delta_p v_n \rightarrow -\Delta_p v$ in $D^{1,p}(\Omega)$ and for g locally lipchitz continuous in $(0, \infty)$ we have $g(v_n) \rightarrow g(v)$ pointwise.

Hence using Lebesgue Dominated Convergence Theorem and the fact that g is increasing on $(0, \infty)$, we have,

$$\int_{\Omega} |\nabla u|^p \geq \int_{\Omega} \frac{|u|^p}{g(v)} (-\Delta_p v)$$

for any $u \in C_c^\infty(\Omega)$. □

Before we proceed with the proof of Theorem 2.4, we will state the Strong Maximum Principle from [11] which was proved using the Harnack Inequality of [16].

Lemma 2.7. (Strong Maximum Principle) *Let $p > 1$ and $\Omega \subset \mathcal{H}^n$ be a bounded domain and $u \in D_0^{1,p}(\Omega)$ be nonnegative solution of the following equation*

$$-\Delta_p u = h(x, u) \text{ in } \Omega; \quad u|_{\partial\Omega} = 0$$

where $h : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ is a measurable function such that $|h(x, u)| \leq C(u^{p-1} + 1)$. Then $u \equiv 0$ or $u > 0$ in Ω .

With Lemma 2.6 and Lemma 2.7 in hand we now proceed with the proof of the Theorem 2.4.

Proof. Using the Strong Maximum Principle we have $v > 0$ in Ω . Denote, $v_n(x) = v(x) + \frac{1}{n}$, $n \in \mathbb{N}$. Thus we have the following:

- $\Delta_p v_m = \Delta_p v$.
- $v_n \rightarrow v$ a.e in Ω and in $D^{1,p}(\Omega)$.
- $g(v_n) \rightarrow g(v)$ a.e in Ω .

Hence using Lemma 2.6 we have for $u \in C_c^\infty(\Omega)$,

$$\int_{\Omega} |\nabla u|^p \geq \int_{\Omega} \frac{|u|^p}{g(v)} (-\Delta_p v)$$

Now to conclude our theorem for $u \in D_0^{1,p}(\Omega)$, we use $u_n \in C_0^\infty(\Omega)$ such that $u_n \rightarrow u$ in $D_0^{1,p}(\Omega)$. Choosing u_n and v_n in Lemma 2.6, we have

$$\int_{\Omega} |\nabla u_n|^p \geq \int_{\Omega} \left(\frac{|u_n|^p}{g(v_n)} \right) (-\Delta_p v_n)$$

Now using the fact that g satisfies (2.1) we have by Fatou Lemma,

$$\int_{\Omega} |\nabla u|^p \geq \int_{\Omega} \left(\frac{|u|^p}{g(v)} \right) (-\Delta_p v)$$

which concludes our proof. \square

We will conclude this section with the Díaz-Saá Inequality in Heisenberg Group. For this part we will be using $g(u) = u^{p-1}$.

Theorem 2.8. (*Díaz-Saá Inequality*) *Let $p > 1$ and Ω be a bounded domain in \mathcal{H}^n . If $u_i \in D_0^{1,p}(\Omega)$ s.t $-\Delta_p u_i = \mu_i$, where $\mu_i > 0$ is a bounded Radon measure with $u_i|_{\partial\Omega} = 0$ and $u_i(\not\equiv 0) \geq 0$ a.e in Ω for $i = 1, 2$. Then we have,*

$$\int_{\Omega} \left(-\frac{\Delta_p u_1}{u_1^{p-1}} + \frac{\Delta_p u_2}{u_2^{p-1}} \right) (u_1^p - u_2^p) \geq 0.$$

Remark 2.9. *Note that above theorem is not true for a general g satisfying (2.1).*

Proof. Choosing u_i for $i = 1, 2$ satisfying the hypothesis of Theorem 2.8 and then plugging the tuple (u_1, u_2) into Theorem 2.4 we get

$$\int_{\Omega} |\nabla u_1|^p \geq \int_{\Omega} \left(-\frac{\Delta_p u_2}{u_2^{p-1}} \right) u_1^p$$

Using Integration by Parts on right part, we have

$$\int_{\Omega} \left(-\frac{\Delta_p u_1}{u_1^{p-1}} + \frac{\Delta_p u_2}{u_2^{p-1}} \right) u_1^p \geq 0. \quad (2.10)$$

Now interchanging the tuple (u_1, u_2) into (u_2, u_1) in Theorem 2.4 we get,

$$\int_{\Omega} |\nabla u_2|^p \geq \int_{\Omega} \left(-\frac{\Delta_p u_1}{u_1^{p-1}} \right) u_2^p$$

Again using Integration by Parts on the right part we have,

$$\int_{\Omega} \left(-\frac{\Delta_p u_2}{u_2^{p-1}} + \frac{\Delta_p u_1}{u_1^{p-1}} \right) u_2^p \geq 0. \quad (2.11)$$

Adding (2.10) and (2.11) we have,

$$\int_{\Omega} \left(-\frac{\Delta_p u_1}{u_1^{p-1}} + \frac{\Delta_p u_2}{u_2^{p-1}} \right) (u_1^p - u_2^p) \geq 0.$$

Hence proved. \square

3. MAIN RESULTS

In this section of the paper we will state and proof our main result.

Theorem 3.1. (*Uniqueness of Solution*) *There exists at most one positive weak solution to equation (2.1) in $D_0^{1,p}(\Omega)$.*

Proof. Let u and v be two non-positive solutions of equation (1.1). Then using Lemma 2.7 we have, $u, v > 0$ in Ω . Moreover since $f(x, u)$ is positive and satisfy hypothesis (I), we have for $u \neq v$,

$$0 \geq \int_{\Omega} \left(-\frac{\Delta_p u}{u^{p-1}} + \frac{\Delta_p v}{v^{p-1}} \right) (u^p - v^p) = \int_{\Omega} \left(\frac{f(x, u)}{u^{p-1}} - \frac{f(x, v)}{v^{p-1}} \right) (u^p - v^p) < 0.$$

Hence we arrive at a contradiction. \square

We now proceed to the proof of existence of solution in $D_0^{1,p}(\Omega)$. The energy functional corresponding to equation (1.1) as

$$E(u) = \frac{1}{p} \int_{\Omega} |\nabla u|^p - \int_{\Omega} F(x, u) \text{ with } u \in D_0^{1,p}(\Omega)$$

where $F(x, u) = \int_0^u f(x, t) dt$.

Theorem 3.2. (*Existence of Solution*) *The equation (2.1) admits a solution if the following two conditions hold simultaneously:*

$$\lambda_1(-\Delta_p v - a_o |v|^{p-2} v) < 0 \text{ with } a_o(x) = \lim_{r \searrow 0} \frac{f(x, r)}{r^{p-1}} \quad (3.1)$$

$$\lambda_1(-\Delta_p v - a_{\infty} |v|^{p-2} v) > 0 \text{ with } a_{\infty}(x) = \lim_{r \nearrow \infty} \frac{f(x, r)}{r^{p-1}}. \quad (3.2)$$

Using the exact same proof of [3] (Théorème 2), one can show the existence of the minimizer to $E(u)$ in $D_0^{1,p}(\Omega)$ and hence Theorem 3.2.

COMMENTS

We conclude this article with a few comments:

- It is worth mentioning again that the methods used in Díaz-Saá [3] for proving the existence and uniqueness of equation (1.1) can't be used here due to non-availability of the $C^{1,\alpha}$ regularity of the p -sub-Laplacian in Heisenberg Group for all $p > 1$, so we have used the Picone Inequality to bypass the problem but in doing so we are forced to put the positivity condition on f , which was not present in the assumptions of [3]. It will be interesting to know if one can conclude the same results for uniqueness without the positivity condition on f .
- The statements proved in Theorem 2.2 and Theorem 2.4 were valid for a wide range of functions satisfies (2.1). This results are new in the contexts of Heisenberg group and one can actually obtain Nonexistence results and Comparison Principles for p -sub-Laplacian similar to those in [14] and the reference therein.

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